On Dwell Time Minimization for Switched Delay Systems: Time-Scheduled Lyapunov Functions

Ahmet Taha Koru*•, Akın Delibaşı•, Hitay Özbay

Yıldız Technical University
Bilkent University
Section 1

Introduction
What is a switched (hybrid) system?

Includes two kinds of behaviour

- A set of dynamic systems
  - Continuous or discrete
  - Linear or nonlinear
- A discrete switching event

General behaviour of a switched system

\[ \dot{x}(t) = f_{\sigma(t)}(x(t)), \quad \forall t > t_0 \]
Switching signals

- Piecewise constant
- A mapping from time to an index set, $\sigma : \mathbb{R} \to \mathcal{P}$
  - Index set represents the subsystems
  - Finite number of subsystems, $\mathcal{P} = \{1, 2, \ldots, m\}$
Stability analysis approaches for switched systems: Arbitrary switching

- All subsystems are stable
  - Short proof: Suppose system 1 is not stable. $\sigma(t) = 1$, $\forall t$
- A popular condition: Common Lyapunov function

![Graph showing Lyapunov function over time](image)

Ahmet Taha Koru $^{a,b}$, Akın Delibaşı $^a$, Hitay Özbay $^b$

$^a$Yıldız T.U., $^b$Bilkent U.

Dwell Time Minimization
Stability analysis approaches for switched systems: Slow switching I

- All subsystems are stable
- Piecewise Lyapunov function
- Decreasing sequence: $V(t_1) > V(t_2) > \ldots$

![Graph showing piecewise Lyapunov function with decreasing sequence over time.](image)
Stability analysis approaches for switched systems: Slow switching II

Definition

Lin, Antsaklis (2009). A positive constant $\tau_D \in \mathbb{R}$ is called the dwell time of a switching signal if the time interval between any two consecutive switchings is no smaller than $\tau_D$.

Stability analysis: Determining an upper bound for the dwell time!
Slow switching stability analysis methods

- **Method I: Exponential stability**
  - $V_i(t) \leq e^{-\alpha t} V_i(t_0)$, $V_i(t) \leq \mu V_j(t)$, $\tau_D = \ln(\mu)/\alpha$
  - Advantages: Low conservativeness
  - Disadvantages: Minimizing dwell time is not convex!

- **Method II: Time-scheduled Lyapunov functions**
  - Discretize the conditions: $V_i(t) = x^T(t) P(t) x(t)$, $P_i(\tau_D) > P_j(t_0)$
  - Advantages: Low conservativeness, linear semi-definite programming
  - Disadvantages: High computational cost

- The other methods...
Our method

We analyzed dwell time based stability of switched delay systems via time-scheduled Lyapunov functions.
Why are slow switching strategies popular?

Comparison with arbitrary switching.

- Not all of switching systems share a common Lyapunov function
- Less conservativeness, more performance
- To avoid fast switching

- Real life examples have dwell time. Examples:
  - Changing road condition of a car (dirt, wet, dry)
  - A surgical robot touching a tissue or not
  - Gear systems
Section 2

Preliminaries and Problem Definition
The system under consideration

Switched delay systems

\[ \dot{x}(t) = A_{\sigma(t)} x(t) + \bar{A}_{\sigma(t)} x(t - r(t)), \quad t \geq 0 \]
\[ x(\theta) = \varphi(\theta), \quad \forall \theta \in [-\tau_{\text{max}}, 0] \]

Properties of the subsystems

- Linear systems
- Time-varying delay
  - \[ |\dot{r}_{\sigma(t)}| \leq d \]
  - \[ 0 < r_{\sigma(t)} < h \]
To ease the notation

We introduce the double:

$$\Sigma_i := (A_i, \tilde{A}_i) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}$$

to denote $i^{th}$ subsystem and get rid of the $\sigma(t)$ notation.
Free weighting matrices method

**Lemma**

For the free parameters $P_i \succeq 0$, $Q_i \succeq 0$, $Z_i \succeq 0$, $X_i \succeq 0$

$$\phi_i = \begin{bmatrix} \phi_{11i} & \phi_{12i} & hA_i^T Z_i \\ * & \phi_{22i} & h\bar{A}_i^T Z_i \\ * & * & -h_i Z_i \end{bmatrix} \prec 0 \quad \psi_i = \begin{bmatrix} X_{11i} & X_{12i} & N_{1i} \\ * & X_{22i} & N_{2i} \\ * & * & Z_i \end{bmatrix} \succeq 0$$

$$\phi_{11i} = P_i A_i + A_i^T P_i + N_{1i} + N_{1i}^T + Q_i + hX_{11i},$$
$$\phi_{12i} = P_i \bar{A}_i - N_{1i} + N_{2i}^T + hX_{12i},$$
$$\phi_{22i} = -N_{2i} - N_{2i}^T - (1 - d) Q_i + \tau_i X_{22i}.$$

non-switching $\Sigma_i$ is stable.
What to do with those conditions?

- Extend free weighting matrices method to Time-scheduled Lyapunov functions for non-switching case
- Find the conditions of stability of switched delay systems under dwell time constraints
- Minimize the upper bound of the dwell time
Section 3

Main Results
Time-scheduled Lyapunov function I

\[ V_i(t, x_t) = x^T(t)P_i(t)x(t) + \int_{t-r(t)}^{t} x^T(s)Q_i(t)x(s)ds \]

\[ + \int_{t-r(t)}^{t} (s - t + h)x^T(s)\dot{Q}_i(t)x(s)ds \]

\[ + \int_{-h}^{0} \int_{0}^{t} \dot{x}(s)Z_i(t)\dot{x}(s)dsd\theta \]

\[ + \int_{-h}^{0} \int_{0}^{t} (s - t - \theta)\dot{x}(s)\dot{Z}_i(t)\dot{x}(s)dsd\theta \]
Time-scheduled Lyapunov function II

\[ P_i(t) = \begin{cases} \text{affine}(P_{i,k}, P_{i,k+1}, t_{j,k}, \delta_k), & t \in \Omega \\ P_{i,K} & t \in [t_{j,k}, t_{j+1}] \end{cases} \]

Similar to \( P_i(t) \), \( Q_i(t) \) and \( Z_i(t) \) affine functions.

There are \( K \) many affine segments.

\[ \Omega = \{ t \mid t \in [t_{j,k}, t_{j,k+1}], \forall k = 0, 1, \ldots, K - 1 \} \]
The equality $x^T P x = 1$ defines an ellipsoid.

Not safe to switch

$P_1 \preceq P_2$

Safe to switch, $1 \rightarrow 2$.

$P_2 \succeq P_1$

Safe to switch, $2 \rightarrow 2$. 
As a result, you can switch after the dwell time $\tau_D$.
On the road to the dwell time

- **Non-switching stability**: Time-scheduled Lyapunov functions must be decreasing for one segment, i.e., $P_i(t) \uparrow$, $x^T P_i(t) x \downarrow$

- **Switching stability**: At the final point, the Lyapunov functions of the active system must be greater than all of other systems’ initial Lyapunov functions, i.e., $P_i(t) \geq P_j(t_0)$, $\forall t \geq \tau_D$, $\forall i, j$
Stability Conditions

For given $\tau_D$ and $K$, feasibility problem is semi-definite programming

\begin{align*}
1\phi_{i,k} &< 0, & 2\phi_{i,k} &< 0, & \forall k = 0, \ldots, K - 1 \\
\psi_{i,k} &\geq 0, & \forall k = 0, \ldots, K \\
\phi_{i,K} &< 0, & \\
Q_{i,k+1} - Q_{i,k} &\geq 0, & \forall k = 0, \ldots, K - 1 \\
Z_{i,k+1} - Z_{i,k} &\geq 0, & \forall k = 0, \ldots, K - 1 \\
P_{i,K} - P_{j,0} &\geq 0, & \forall j \in \mathcal{P}, j \neq i \\
Q_{i,K} - Q_{j,0} - \frac{h}{\delta} (Q_{j,1} - Q_{j,0}) &\geq 0, & \forall j \in \mathcal{P}, j \neq i \\
Z_{i,K} - Z_{j,0} - \frac{h}{\delta} (Z_{j,1} - Z_{j,0}) &\geq 0, & \forall j \in \mathcal{P}, j \neq i
\end{align*}
Feasibility problem is convex for fixed $\tau_D$, however dwell time minimization problem is quasi-convex.

*Bisection algorithm* generates a sequence of SDP.
Section 4

Numerical Examples
An example

$\Sigma_1$:

\[ A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad \bar{A}_1 = \begin{bmatrix} -1 & 0 \\ -0.5 & -1 \end{bmatrix}, \]

$\Sigma_2$:

\[ A_2 = \begin{bmatrix} -1 & 0.5 \\ 0 & -1 \end{bmatrix}, \quad \bar{A}_2 = \begin{bmatrix} -1 & 0 \\ 0.1 & -1 \end{bmatrix}, \]

\[ h = 0.6 \text{ seconds}, \quad d = 0. \]
Comparison of the presented method

Table: Comparison of Example I and II

<table>
<thead>
<tr>
<th>Ex.</th>
<th>Yan</th>
<th>Çalışkan</th>
<th>Koru</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.51 s</td>
<td>3.4 s</td>
<td>1.11 s</td>
<td>1.06×10^{-5} s</td>
</tr>
<tr>
<td>2</td>
<td>–</td>
<td>0.72 s</td>
<td>0.58 s</td>
<td>7.26×10^{-6} s</td>
</tr>
</tbody>
</table>
Section 5

Conclusions
Dwell time stability conditions for switched delay systems are presented.

The performance of the method is shown with numerical examples.
THANK YOU