On Dwell Time Minimization for Switched Delay Systems: Time-Scheduled Lyapunov Functions

Ahmet Taha Koru*[,a], Akın Delibaşı[a], Hitay Özbay[b]

Yıldız Technical University
Bilkent University

Section 1

Introduction

What is a switched (hybrid) system?

Includes two kinds of behaviour

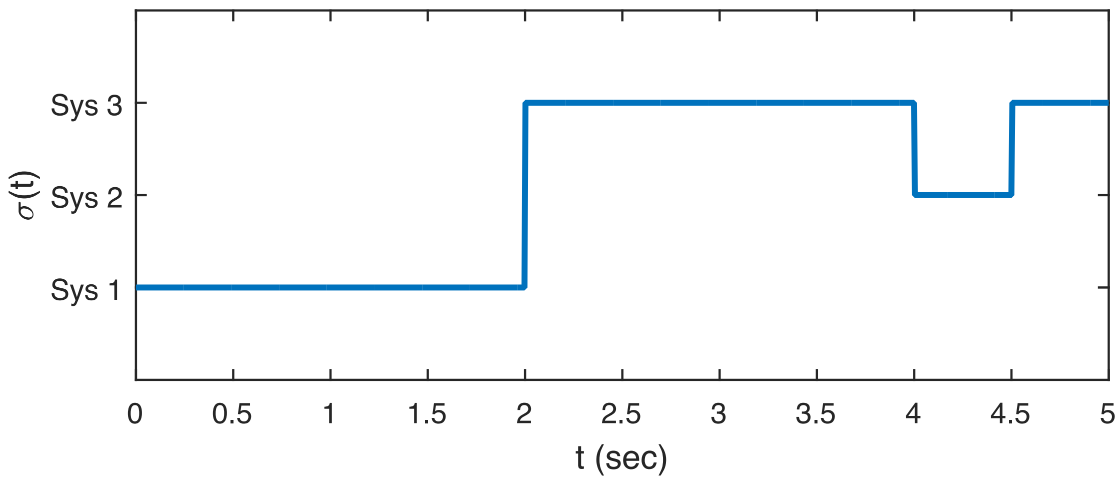
- A set of dynamic systems
  - Continuous or discrete
  - Linear or nonlinear
- A discrete switching event

General behaviour of a switched system

$$\dot{x}(t) = f_{\sigma(t)}\left(x(t)\right), \quad \forall t > t_0$$

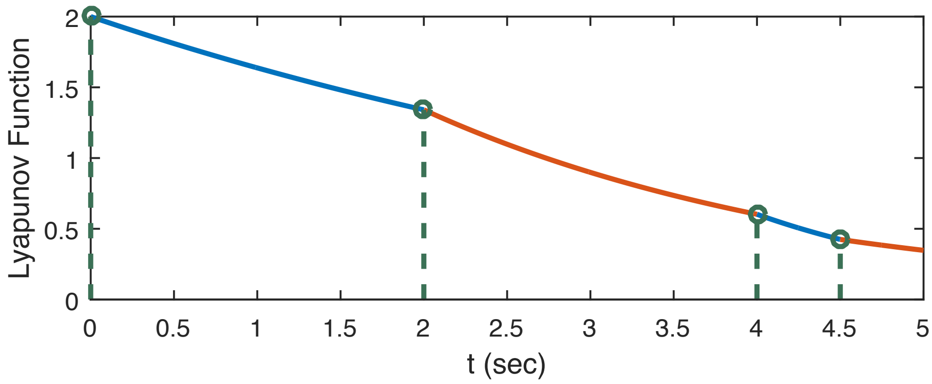
Switching signals

- Piecewise constant
- A mapping from time to an index set, $\sigma : \mathbb{R} \to \mathcal{P}$
  - Index set represents the subsystems
  - Finite number of subsystems, $\mathcal{P} = \{1, 2, \ldots, m\}$



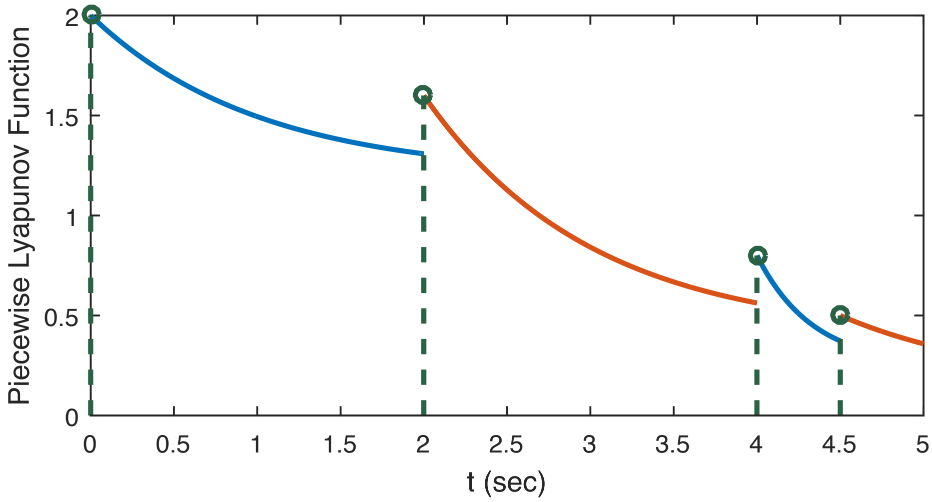
Stability analysis approaches for switched systems: Arbitrary switching

- All subsystems are stable
  - Short proof: Suppose system 1 is not stable. $\sigma(t) = 1, \quad \forall t$
- A popular condition: Common Lyapunov function



Stability analysis approaches for switched systems: Slow switching I

- All subsystems are stable
- Piecewise Lyapunov function
- Decreasing sequence: $V(t_1) > V(t_2) > \dots$



Stability analysis approaches for switched systems: Slow switching II

Definition

Lin, Antsaklis(2009). A positive constant $\tau_D \in \mathbb{R}$ is called the dwell time of a switching signal if the time interval between any two consecutive switchings is no smaller than τ_D.

Stability analysis: Determining an upper bound for *the dwell time*!

Slow switching stability analysis methods

- Method I: Exponential stability
  - $V_i(t) \leq e^{-\alpha t} V_i(t_0)$, $V_i(t) \leq \mu V_j(t)$, $\tau_D = \ln(\mu)/\alpha$
  - Advantages: Low conservativeness
  - Disadvantages: Minimizing dwell time is not convex!
- Method II: Time-scheduled Lyapunov functions
  - Discretize the conditions: $V_i(t) = x^T(t)P(t)x(t)$, $P_i(\tau_D) > P_j(t_0)$
  - Advantages: Low conservativeness, linear semi-definite programming
  - Disadvantages: High computational cost
- The other methods...

Our method

We analyzed dwell time based stability of switched delay systems via time-scheduled Lyapunov functions.

Why are slow switching strategies popular?

Comparison with arbitrary switching.

- Not all of switching systems share a common Lyapunov function
- Less conservativeness, more performance
- To avoid fast switching
- Real life examples have dwell time. Examples:
  - Changing road condition of a car (dirt, wet, dry)
  - A surgical robot touching a tissue or not
  - Gear systems

Section 2

Preliminaries and Problem Definition

The system under consideration

Switched delay systems

$$\dot{x}(t) = A_{\sigma(t)}x(t) + \bar{A}_{\sigma(t)}x(t - r(t)), \quad t \geq 0$$
$$x(\theta) = \varphi(\theta), \quad \forall\theta \in [-\tau_{\max}, 0]$$

Properties of the subsystems

- Linear systems
- Time-varying delay
  - $|\dot{r}_{\sigma(t)}| \leq d$
  - $0 < r_{\sigma(t)} < h$

To ease the notation

We introduce the double:

$$\Sigma_i := \left(A_i, \bar{A}_i\right) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}$$

to denote $i^t h$ subsystem and get rid of the $\sigma(t)$ notation.

Free weighting matrices method

Lemma

For the free parameters $P_i \succeq 0$, $Q_i \succeq 0$, $Z_i \succeq 0$, $X_i \succeq 0$

$$\phi_i = \begin{bmatrix} \phi_{11i} & \phi_{12i} & hA_i^T Z_i \\ * & \phi_{22i} & h\bar{A}_i^T Z_i \\ * & * & -h_i Z_i \end{bmatrix} \prec 0 \quad \psi_i = \begin{bmatrix} X_{11i} & X_{12i} & N_{1i} \\ * & X_{22i} & N_{2i} \\ * & * & Z_i \end{bmatrix} \succeq 0$$

$$\phi_{11i} = P_i A_i + A_i^T P_i + N_{1i} + N_{1i}^T + Q_i + hX_{11i},$$
$$\phi_{12i} = P_i \bar{A}_i - N_{1i} + N_{2i}^T + hX_{12i},$$
$$\phi_{22i} = -N_{2i} - N_{2i}^T - (1-d)Q_i + \tau_i X_{22i}.$$

non-switching Σ_i is stable.

What to do with those conditions?

- Extend free weighting matrices method to Time-scheduled Lyapunov functions for non-switching case
- Find the conditions of stability of switched delay systems under dwell time constraints
- Minimize the upper bound of the dwell time

Section 3

Main Results

Time-scheduled Lyapunov function I

$$V_i(t,x_t) = x^T(t)P_i(t)x(t) + \int_{t-r(t)}^{t} x^T(s)Q_i(t)x(s)ds$$

$$+ \int_{t-r(t)}^{t} (s-t+h)\,x^T(s)\dot{Q}_i(t)x(s)ds$$

$$+ \int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}(s)Z_i(t)\dot{x}(s)dsd\theta$$

$$+ \int_{-h}^{0}\int_{t+\theta}^{t} (s-t-\theta)\,\dot{x}(s)\dot{Z}_i(t)\dot{x}(s)dsd\theta$$

Time-scheduled Lyapunov function II

$$P_i(t) = \begin{cases} \text{affine}(P_{i,k}, P_{i,k+1}, t_{j,k}, \delta_k), & t \in \Omega \\ P_{i,K} & t \in \left[t_{j,K}, t_{j+1}\right) \end{cases}$$

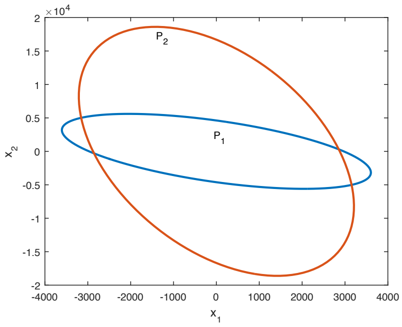
Similar to $P_i(t)$, $Q_i(t)$ and $Z_i(t)$ affine functions.

There are K many affine segments.

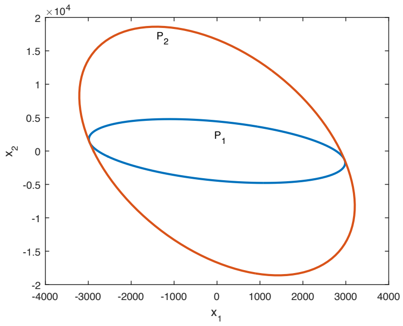
$$\Omega = \left\{ t \,|\, t \in \left[t_{j,k},\ t_{j,k+1} \right),\ \forall k = 0, 1, \ldots, K-1 \right\}$$

Visualization of what's going on

The equality $x^T P x = 1$ defines an ellipsoid.

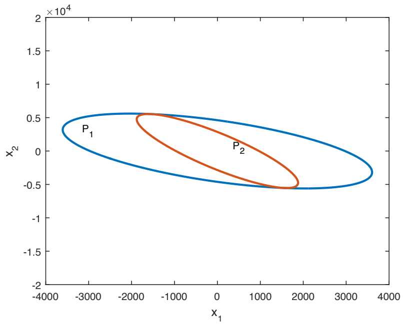


Not safe to switch



$P_1 \succeq P_2$

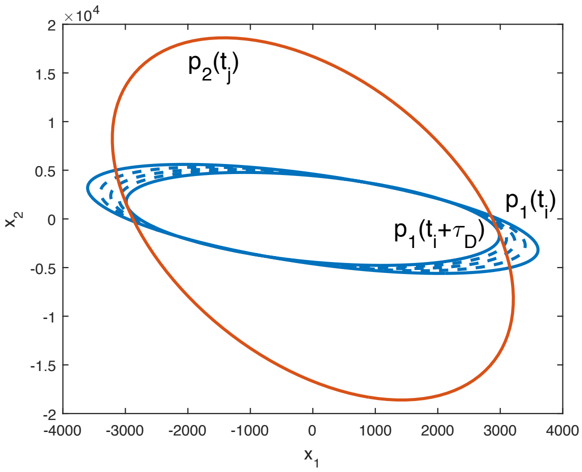
Safe to switch,1 $\rightarrow$ 2.

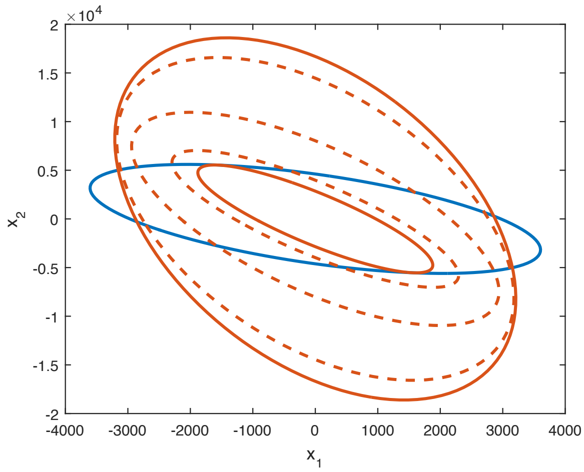


$P_2 \succeq P_1$

Safe to switch, 2 $\rightarrow$ 2.

Visualization of time-scheduled Lyapunov functions





$$P_1(t) \succeq P_2(t_0), \quad t \geq t_0 + \tau_D \qquad\qquad P_2(t) \succeq P_1(t_0), \quad t \geq t_0 + \tau_D$$

As a result, you can switch after the dwell time τ_D.

On the road to the dwell time

- **Non-switching stability:** Time-scheduled Lyapunov functions must be decreasing for one segment, i.e., $P_i(t) \uparrow, \quad x^T P_i(t) x \downarrow$

- **Switching stability:** At the final point, the Lyapunov functions of the active system must be greater than all of other systems' initial Lyapunov functions, i.e., $P_i(t) \succeq P_j(t_0), \quad \forall t \geq \tau_D, \quad \forall i, j$

Stability Conditions

For given τ_D and K, feasibility problem is semi-definite programming

$$
\begin{aligned}
&{}^{1}\phi_{i,k} \prec 0, \quad {}^{2}\phi_{i,k} \prec 0, && \forall k = 0, \ldots, K-1 \\
&\psi_{i,k} \succeq 0, && \forall k = 0, \ldots, K \\
&\phi_{i,K} \prec 0, && \\
&Q_{i,k+1} - Q_{i,k} \succeq 0, && \forall k = 0, \ldots, K-1 \\
&Z_{i,k+1} - Z_{i,k} \succeq 0, && \forall k = 0, \ldots, K-1 \\
&P_{i,K} - P_{j,0} \succeq 0, && \forall j \in \mathcal{P},\ j \neq i \\
&Q_{i,K} - Q_{j,0} - \frac{h}{\delta}\left(Q_{j,1} - Q_{j,0}\right) \succeq 0, && \forall j \in \mathcal{P},\ j \neq i \\
&Z_{i,K} - Z_{j,0} - \frac{h}{\delta}\left(Z_{j,1} - Z_{j,0}\right) \succeq 0, && \forall j \in \mathcal{P},\ j \neq i
\end{aligned}
$$

Minimization of the dwell time

Feasibility problem is convex for fixed τ_D, however dwell time minimization problem is quasi-convex.

Bisection algorithm generates a sequence of SDP.

Section 4

Numerical Examples

An example

Σ_1:

$$A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad \bar{A}_1 = \begin{bmatrix} -1 & 0 \\ -0.5 & -1 \end{bmatrix},$$

Σ_2:

$$A_2 = \begin{bmatrix} -1 & 0.5 \\ 0 & -1 \end{bmatrix}, \quad \bar{A}_2 = \begin{bmatrix} -1 & 0 \\ 0.1 & -1 \end{bmatrix},$$

$h = 0.6$ seconds, $d = 0$.

Comparison of the presented method

Table: Comparison of Example I and II

Ex.	Yan	Çalışkan	Koru	Present
1	6.51 s	3.4 s	1.11 s	1.06×10^{-5} s
2	–	0.72 s	0.58 s	7.26×10^{-6} s

Section 5

Conclusions

- Dwell time stability conditions for switched delay systems are presented
- The performance of the method is shown with numerical examples

THANK YOU