$\mathcal{L}_2$ Control of LPV Systems with Saturating Actuators: Pólya Approach

Akın Delibaşı, Ibrahim B. Kucukdemiral and Galip Cansever

Abstract

This paper addresses the design problem of $\mathcal{L}_2$, gain scheduling nonlinear state feedback controller for Linear Parameter Varying (LPV) systems, subjected to actuator saturations and bounded energy disturbances, by using parameter dependent Lyapunov functions. The paper provides a systematic procedure to generate a sequence of Linear Matrix Inequality (LMI) conditions of increasing precision for obtaining a sub-optimal $\mathcal{L}_2$ state-feedback controller. The presented method utilizes the modified sector condition for actuator saturation formalization and homogeneous-polynomial-parameter-dependent (HPPD) representation of LPV systems. Both simulation and experimental studies on an inverted pendulum system illustrate the benefits of the approach.

keywordsHomogeneous-Polynomial-Parameter-Dependent Lyapunov Functions, Input to State Stability:LMIs, Actuator Saturations

1 Introduction

Gain-scheduling control is one of the most popular and promising nonlinear control methods which gathers the simple structure of linear controllers together with the global effectiveness of the nonlinear controllers. Therefore, in the last two decades, this method has been widely and successfully applied in fields ranging from chemical process control to aerospace systems. The survey articles [?, ?] provide a good account for the current state of research in the area. This technique is also a very powerful tool for controlling linear parameter varying (LPV) systems. LPV control technique has certain advantages over other nonlinear techniques. For example, it is adaptable to system parameter variations, it does not require severe structural assumptions on the plant model and the approach can be used in the absence of complete analytical plant models. Besides, it can be extended to more complex control problems easily and gives the opportunity to use different performance metrics such as performance and robustness in the control problem. The last decade has witnessed significant advances in the theory of LPV systems and their control. Especially, as being different from the early methods which utilize parameter independent Lyapunov functions, the recent techniques are mostly based on parameter-dependent Lyapunov functions [?, ?, ?, ?, ?]. As is known, using parameter-dependent Lyapunov functions instead of non-parametric ones, remarkably decrease the conservatism of the resulting controllers.

In recent years, some of these Lyapunov-based methods, originally presented for the robust stability issue, have also been extended to cope with robust performance problems. For instance, guaranteed $\mathcal{H}_\infty$ and $\mathcal{H}_2$ control problems are mostly concerned. Some of the recent methods rely on affine-parameter-dependent methods [?, ?, ?], methods utilizing polynomial-parameter-dependent matrices [?, ?] and finally the methods that are based on homogeneous-polynomial-parameter-dependent (HPPD) Lyapunov functions [?, ?, ?, ?].

It is well known that, in a control system, one of the main restrictions that affects the overall system performance and stability is actuator saturation. Saturations in actuators are important nonlinearities which may effect the overall stability of the closed loop systems. For instance, in recent years, various studies on actuator saturations have been made such as [?, ?, ?, ?, ?, ?, ?, ?]. Among them, the results of [?] attract attention, since their method provides a finite dimensional convex condition to design HPPD state feedback control gains which ensure stability in a region of the state space for linear systems affected by parameters that can vary arbitrarily fast inside a polytope. However, this method is based on parameter-independent Lyapunov functions which is still conservative.

The focus of this paper is on techniques to obtain a new systematic procedure to generate a sequence of LMI conditions of increasing precision for obtaining a sub-optimal state-feedback gain-scheduling controller for LPV systems subject to actuator saturations and $\mathcal{L}_2$-bounded disturbances which ensures $\mathcal{L}_2$-gain minimization of the closed-loop system from disturbance input to the controlled outputs. The proposed method utilizes the modified sector condition which has been recently used in [?]. Mainly, our technique is based on two particular tools: First, our method is based on the homogeneous polynomially parameter dependent Lyapunov functions which drastically reduce the conservatism of the system when compared with the non-parametric Lyapunov methods and secondly, it is based on modified sector condition for actuator saturation description. By use of the proposed method, a systematic procedure for generating a family of LMI conditions that depends on polynomial Pólya relaxation level d and polynomial degree (g) are obtained. The method provides that as g and/or d tends to infinity, less and less conservative results are obtained. This result also confirmed through simulations and laboratory experiments on an inverted pendulum benchmark system.

The notation to be used in the paper is fairly standard: $\mathbb{R}$ stands for the set of real numbers, the set of square integrable vector functions with dimension q is denoted by L_2^q, sat stands for the well-known saturation function. $\mathbb{R}^{m\times n}$ is the set of $m \times n$ dimensional real matrices. $diag$ denotes the diagonal matrices, Tr stands for the trace operator, $\mathbb{Z}_+$ symbolizes the set of positive integers. The identity and null matrices are denoted by I and 0, respectively. $X > 0(\geq, < 0)$ denotes that X is a positive definite (positive semi-definite, negative definite) matrix. $X > Y$ means that $X - Y$ is positive definite. Finally, the notation $\star$ denotes off diagonal block completion of a symmetric matrix. Note that in order to lighten the notation, we use the standard convention of dropping the time argument from the variables in time-varying variables.

Rest of the paper is organized as follows: Section 2 states the problem formulation, Section 3 presents the mathematical preliminaries, the main results of the paper is presented in Section 4, Section 5 provides simulation and experimental results on an inverted pendulum system. Finally Section 6 provides concluding remarks.

2 Problem formulation

Consider the system

$$\begin{aligned} \dot{x} &= A(\alpha)x + B_w(\alpha)w + B_u u \\ z &= C(\alpha)x \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control vector, $w \in L_2^q[0,\infty)$ is an exogenous disturbance signal with limited in energy, i.e. $\exists\, \delta$, $0 < 1/\delta < \infty$, such that

$$\|w\|_2^2 = \int_0^\infty w(\tau)^T w(\tau) d\tau < \frac{1}{\delta}. \tag{2}$$

$z \in \mathbb{R}^p$ is the exogenous controlled output , $\alpha \in \mathbb{R}^N$ is the time varying parameter vector that is assumed to be measurable online and vary in a polytope

$$\Theta = \left\{ \alpha \in \mathbb{R}^N \,:\, \sum_{j=1}^{N} \alpha_j = 1,\, \alpha_j \geq 0,\, j = 1, \ldots, N \right\}. \tag{3}$$

Then the polytope $\mathcal{P}$ can be expressed as

$$\mathcal{P} = \left\{ (A, B_u, B_w, C)(\alpha) : (A, B_u, B_w, C)(\alpha) = \sum_{j=1}^{N} \alpha_j (A, B_u, B_w, C)_j, \alpha \in \Theta \right\}. \tag{4}$$

Besides, we assume that the control signal is saturated in the form of state feedback control law, i.e.,

$$u = sat(K(\alpha)x)\ ,\ K(\alpha) \in \mathbb{R}^{m \times n} \tag{5}$$

where each component of the control vector is $u_i = sat(K_i(\alpha)x)$, for every $i = 1, \ldots, m$ and $\alpha \in \Theta$. Here,

$$u_i = \begin{cases} \rho_i & \text{if} \quad K_i(\alpha)x > \rho_i \\ K_i(\alpha)x & \text{if} \quad -\rho_i \leq K_i(\alpha)x \leq \rho_i \\ -\rho_i & \text{if} \quad K_i(\alpha)x < -\rho_i \end{cases} \tag{6}$$

Then the closed-loop system can be represented as

$$\dot{x} = A_{cl}(\alpha)x - B_u\psi(K(\alpha)x) + B_w(\alpha)w \tag{7}$$

where $A_{cl}(\alpha) := A(\alpha) + B_uK(\alpha)$ and

$$\psi(K(\alpha)x) := K(\alpha)x - sat(K(\alpha)x) \tag{8}$$

Our objective is to compute such a state-feedback gain matrix $K(\alpha)$ and two sets $\mathcal{S}_0$ and $\mathcal{S}_1$ such that the following requirements are satisfied for the closed-loop system (7) for a given δ:

- When $w = 0$, for any initial condition $x(0) \in \mathcal{S}_0$, the system trajectory converges asymptotically to the origin.

- When $w \neq 0$, the closed-loop trajectories remain bounded in $\mathcal{S}_1$ for any initial condition $x(0) \in \mathcal{S}_0$ and for all w satisfying (2). Then, if these conditions hold simultaneously, then one can find scalar constants $\eta > 0$ and $b \geq 0$ such that

$$\|z(t)\|_2^2 \leq \frac{1}{\eta}\|w(t)\|_2^2 + b \quad \forall t > 0 \tag{9}$$

  where b is the bias term regarding the initial conditions.

3 Mathematical Preliminaries

The following lemmas are essential in the development of our results:

Lemma 1: [?, ?] Consider a parameter dependent LMI which is defined as

$$\mathcal{L}(\zeta, \theta) = \mathcal{L}_0(\theta) + \zeta_1 \mathcal{L}_1(\theta) + \cdots + \zeta_M \mathcal{L}_M(\theta) > 0 \tag{10}$$

where $\theta \in \Theta$. In addition to this, assume that $\mathcal{L}_i(\theta),\ i = 0, \ldots, M$ are continuous functions of θ. If one can find a feasible parameter-dependent solution $\zeta(\theta) \in \mathbb{R}^M$ for every θ, such that $\mathcal{L}(\zeta(\theta), \theta) > 0$, then there exists a homogenous polynomial solution $\zeta^\star(\theta) > 0$ such that, for every θ, $\mathcal{L}(\zeta^\star(\theta), \theta) > 0$.

Lemma 1 holds for θ replaced by α. Now, let us define a matrix $G(\alpha) \in \mathbb{R}^{m\times n}$ and a set

$$\mathcal{S}_a(\alpha) = \{x \in \mathbb{R}^n : |[K_i(\alpha) - G_i(\alpha)]x| \leq \rho_i\} \tag{11}$$

where $i = 1, \ldots, m$ for all $\alpha \in \Theta$. Then, for every $x \in \mathcal{S}_a(\alpha)$ with $\psi(K(\alpha)x)$ given by (8) and for all $\alpha \in \Theta$ one can easily conclude that

$$\psi(K(\alpha)x)^T T(\alpha)(\psi(K(\alpha)x) - G(\alpha)x) \leq 0 \tag{12}$$

for any positive definite diagonal matrix $T(\alpha) \in \mathbb{R}^{m\times m}$. The following theorem has been adapted from [?]

Theorem 1: If there exist a positive definite matrix $W(\alpha) \in \mathbb{R}^{n\times n}$, matrices $M(\alpha) \in \mathbb{R}^{m\times n}$, $Y(\alpha) \in \mathbb{R}^{m\times n}$ and diagonal matrix $S(\alpha) \in \mathbb{R}^{m\times m}$ which makes the matrix inequalities,

$$N_1(\alpha) := \begin{bmatrix} W(\alpha) & M_i(\alpha)^T - Y_i(\alpha)^T & M_i(\alpha)^T - Y_i(\alpha)^T \\ \star & \frac{1}{\beta}\rho_i^2 & 0 \\ \star & \star & \eta\delta\rho_i^2 \end{bmatrix} \geq 0, i = 1, \ldots, m \tag{13}$$

$$N_2(\alpha) := \begin{bmatrix} \mathcal{V}(\alpha) & -B_u S(\alpha) + Y(\alpha)^T & \eta B_w(\alpha) & Q(\alpha)^T C(\alpha)^T \\ \star & -2S(\alpha) & 0 & 0 \\ \star & \star & -\eta I & 0 \\ \star & \star & \star & -I \end{bmatrix} < 0, \tag{14}$$

feasible for every $\alpha \in \Theta$ where

$$\mathcal{V}(\alpha) = A(\alpha)W(\alpha) + W(\alpha)A^T(\alpha) + B_u M(\alpha) + M(\alpha)^T B_u^T + dP(\alpha)/dt \tag{15}$$

then the control rule

$$K(\alpha) = M(\alpha)[W(\alpha)]^{-1} \tag{16}$$

ensures that:

- the closed-loop system is $\mathcal{L}_2-$ stable with a finite gain $\|z(t)\|_2^2 \leq (1/\eta)\|w(t)\|_2^2 + \beta$, $\forall t > 0$ and for any $x(0) \in \mathcal{S}_0 = \mathcal{E}(P(\alpha), \beta) = \{x \in \mathbb{R}^n; x^T P(\alpha) x \leq \beta\}$ and for all $w(t)$ satisfying (2).

- For every $w \neq 0$ but satisfies (2), the closed-loop trajectories remain bounded in the set $\mathcal{S}_1 = \mathcal{E}(P(\alpha), 1/\mu) = \{x : x^T P(\alpha) x \leq 1/\mu\}$ where $1/\mu = \beta + 1/\eta\delta$, for any $x(0) \in \mathcal{S}_0 = \mathcal{E}(P(\alpha), \beta)$.

- when $w = 0$, the set $\mathcal{S}_1 = \mathcal{E}(P(\alpha), 1/\mu)$ is contractive and remains in the basin of attraction of the closed-loop system.

Proof: Consider the Lyapunov function candidate $V(x) = x^T P(\alpha)x$, where $P(\alpha)^T = P(\alpha)$. Pre- and post multiplying the LMIs in (13) by $diag\{[W(\alpha)]^{-1}, 1, 1\}$ and making the variable transformations $M(\alpha) = K(\alpha)W(\alpha)$, $Y(\alpha) = G(\alpha)W(\alpha)$ and $[W(\alpha)]^{-1} = P(\alpha)$, one can readily obtain

$$\begin{bmatrix} P(\alpha) & K_{sat_i}(\alpha)^T - G_i(\alpha)^T & K_{sat_i}(\alpha)^T - G_i(\alpha)^T \\ \star & \frac{1}{\beta}\rho_i^2 & 0 \\ \star & 0 & \eta\delta\rho_i^2 \end{bmatrix} \geq 0, \tag{17}$$

which ensures that $\mathcal{S}_1(P(\alpha), 1/\mu) \subseteq \mathcal{S}_a(\alpha)$, $\forall\alpha \in \Theta$ under the condition $1/\mu = \beta + 1/\eta\delta$. Furthermore, by this choice of μ, one gets $\mathcal{S}_0 = \mathcal{E}(P(\alpha), \beta) \subseteq \mathcal{E}(P(\alpha), 1/\mu)$. Assume that the condition (14) holds. Using the Schur complement, inequality (14) can be replaced by

$$\begin{bmatrix} \mathcal{V}(\alpha) + W(\alpha)^T C(\alpha)^T C(\alpha)W(\alpha) & -B_u S(\alpha) + Y(\alpha)^T & \eta B_w(\alpha) \\ \star & -2S(\alpha) & 0 \\ \star & \star & -\eta I \end{bmatrix} < 0. \tag{18}$$

Now let us define the exogenous state vector $\xi := [x^T \ \psi(K_{sat}(\alpha)x) \ w]^T$. After some simple algebraic manipulations with the definitions $W(\alpha) = [P(\alpha)]^{-1}$, $S(\alpha) = [T(\alpha)]^{-1}$, $Y(\alpha) = G(\alpha)[P(\alpha)]^{-1}$ one obtains

$$\dot{V}(x) + z^T z - \frac{1}{\eta} w^T w - 2\psi(K(\alpha)x)^T T(\alpha)(\psi(K(\alpha)x) - G(\alpha)x) < 0, \quad T > 0 \tag{19}$$

Furthermore, from Lemma 1, if $x \in \mathcal{S}_a(\alpha)$, relation (19) implies that

$$\dot{V}(x) + z'z - \frac{1}{\eta} w'w < 0 \tag{20}$$

Hence, from (20), provided that $x(0) \in \mathcal{S}_0$, we conclude that

$$V(x(T(\alpha))) \leq (1/\eta)\int_0^T w(t)^T w(t)dt + V(x(0)) \leq (1/\eta)\|w\|_2^2 + \beta \leq 1/\eta\delta + \beta = \frac{1}{\mu}, \quad \forall T(\alpha) > 0. \tag{21}$$

Hence, the system trajectories never leave $\mathcal{S}_1$. Thus, from (13), $\forall T(\alpha) > 0$, $x(T) \in \mathcal{S}_a(\alpha)$ which concludes the proof of the second statement. When $T(\alpha)$ approaches to infinity, we obtain $\|z\|_2^2 < (1/\eta)\|w\|_2^2 + \beta$, which concludes the proof of the first statement. Finally when $w(t) = 0$, one can obtain that $\dot{V}(x) \leq 0$ for all $x(t) \in \mathcal{S}_1$ which proves that $\mathcal{S}_1$ is a contractive set. □

Corollary 1: Since the LMIs in Theorem 1 meets the conditions of Lemma 1, if one can find parameter-dependent matrices $W(\alpha), M(\alpha), Y(\alpha)$ and $S(\alpha)$ that solve Theorem 1, then there always exist HPPD matrices $W_g(\alpha), M_g(\alpha), Y_g(\alpha)$ and $S_g(\alpha)$ of arbitrary degree g that solve Theorem 1.

In the next section, some useful definitions and results are given to construct HPPD matrices of arbitrary degree g that solve Theorem 1 in terms of finite number of LMI conditions.

4 MAIN RESULTS

A homogeneous matrix-polynomial $M_g(\theta)$ of degree g can be generally expressed as

$$M_g(\theta) = \sum_{k \in \mathcal{K}(g)} \alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_N^{k_N} M_k, \quad k = k_1k_2\cdots k_N \tag{22}$$

where $\alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_N^{k_N}$, $\alpha \in \Theta$, $k_i \in \mathbb{Z}_+$, $i = 1, \ldots, N$ are the monomials, and $M_k \in \mathbb{R}^{n\times n}$, $\forall k \in \mathcal{K}(g)$ are matrix valued coefficients to be determined.

Here, by definition, $\mathcal{K}(g)$ is the set of N-tuples obtained as all possible combinations of nonnegative integers k_i, $i = 1, \ldots, N$, such that $k_1 + k_2 + \cdots + k_N = g$. [?] Since the number of vertices in the polytope $\mathcal{P}$ is equal to N, the number of elements in $\mathcal{K}(g)$ can be obtain by

$$J(g) = \frac{(N + g - 1)!}{g!(N - 1)!}.$$

For example, for polynomials of degree $g = 1$ with $N = 4$, the possible values of the partial degrees are $\mathcal{K}(1) = \{0001, 0010, 0100, 1000\}$(so $J(1) = 4$), corresponding to the generic form $M_1(\theta) = \alpha_4 M_{0001} + \alpha_3 M_{0010} + \alpha_2 M_{0010} + \alpha_1 M_{0001}$. For $g = 0$ constant matrices are obtained.

As defined in [?], for N-tuples k and k', $k \geq k'$ is valid if $k_i \geq k'_i$, $i = 1, \ldots, N$. Besides, we assume that summation operation $k + k'$ and subtraction operation $k - k'$ (whenever $k \succ k'$) are all valid in componentwise.Note that, we define N-tuple e_i and the coefficient $\pi(k)$ as

$$e_i = 0 \cdots 0 \underbrace{1}_{i-th} 0 \cdots 0, \qquad \pi(k) := (k_1!)(k_2!) \cdots (k_N!).$$

Theorem 2: There exist symmetric positive definite matrix $W_g(\alpha) \in \mathbb{R}^{n\times n}$, HPPD matrices $M_g(\alpha)$ and $Y_g(\alpha) \in \mathbb{R}^{m\times n}$ and a diagonal HPPD matrix $S_g(\alpha) \in \mathbb{R}^{m\times m}$ for an arbitrary degree g that solve (13) and (14) if and only if, there exist a symmetric positive definite matrix $W_k \in \mathbb{R}^{n\times n}$ matrices M_k and $Y_k \in \mathbb{R}^{m\times n}$, diagonal matrices S_k and $L_k \in \mathbb{R}^{q\times q}$, $k \in \mathcal{K}(g)$ and sufficiently large $d \in \mathbb{Z}_+$

that solve

$$\mathcal{N}_{1_k} := \sum_{\substack{k' \in \mathcal{K}(d) \\ k \succeq k'}} \frac{d!}{\pi(k')} \begin{bmatrix} W_{k-k'} & (M_{k-k'})^T_{(l)} - (Y_{k-k'})^T_{(l)} & (M_{k-k'})^T_{(l)} - (Y_{k-k'})^T_{(l)} \\ \star & \zeta_k \frac{1}{\beta} \rho^2_{(l)} & 0 \\ \star & \star & \zeta_k \eta \delta \rho^2_{(l)} \end{bmatrix} \geq 0,$$

$$l = 1, \ldots, m, \quad \forall k \in \mathcal{K}(g+d) \tag{23}$$

$$\mathcal{N}_{2_k} := \sum_{\substack{k' \in \mathcal{K}(d) \\ k \succeq k'}} \sum_{\substack{i \in \{1,\ldots,N\} \\ k_i \succeq k'_i}} \frac{d!}{\pi(k')} \begin{bmatrix} \mathcal{X} & \mathcal{Y} & \zeta_{k,i} \eta B_{w_i} & W^T_{k-k'-e_i} C^T_i \\ \star & -2S_{k-k'-e_i} & 0 & 0 \\ \star & \star & -\zeta_{k,i} \eta I & 0 \\ \star & \star & \star & -\zeta_{k,i} I \end{bmatrix} < 0,$$

$$\forall k \in \mathcal{K}(g+d+1) \tag{24}$$

with

$$\begin{aligned} \mathcal{X} =& A_i W_{k-k'-e_i} + W^T_{k-k'-e_i} A^T_i + B_u M_{k-k'-e_i} + M^T_{k-k'-e_i} B^T_u \\ \mathcal{Y} =& - B_u S_{k-k'-e_i} + Y^T_{k-k'-e_i}, \quad \zeta_k = g!/\pi(k-k') \\ \zeta_{k,i} =& g!/\pi(k-k'-e_i). \end{aligned}$$

In this case HPPD control gain,

$$K_g(\alpha) = M_g(\alpha)[W_g(\alpha)]^{-1} \tag{25}$$

ensures the asymptotic stability of the closed-loop system in the ellipsoidal region $\mathcal{S}_0$, $\forall \alpha \in \Theta$.

Proof: Notice that, $N_1(\alpha)$ can be written as

$$(\sum_{i=1}^{N} \alpha_i)^d N_1(\alpha) = \sum_{k \in \mathcal{K}(g+d)} \alpha_1^{k_1} \alpha_2^{k_2} \cdots \alpha_N^{k_N} \mathcal{N}_{1_k} \tag{26}$$

when $W(\alpha) = W_g(\alpha)$, $M(\alpha) = M_g(\alpha)$ and $Y(\alpha) = Y_g(\alpha)$. Here, $\mathcal{N}_{1_k}$ is given in (23) for any $d \in \mathbb{Z}_+$ since $(\alpha_1 + \cdots + \alpha_N)^d = 1$. On the other hand, It is obvious that if $\mathcal{N}_{1_k} > 0$, then $N_1(\alpha) > 0$, $\forall \alpha \in \Theta$. In reverse direction, based on $P\acute{o}lya's$ Theorem [?], [?], one has that if $N_1(\alpha) > 0$, then there exists a sufficiently large $d \in \mathbb{Z}_+$ such that all the matrix-valued coefficients $\mathcal{N}_{1_k}$ in (26) are positive definite. By using similar steps, it is easy to show that (24) is equivalent (for sufficiently large $d \in \mathbb{Z}_+$) to the feasibility of (14) and consequently $S_g(\alpha) > 0$, $\forall \alpha \in \Theta$, for a given degree g. Then, based on (16), one has that the control gain is $K_g(\alpha) = M_g(\alpha)[W_g(\alpha)]^{-1}$, as in (25). □

Corollary 2: If there exists a feasible solution to the conditions of Theorem 2 for some $g = \hat{g}$ and $d = \hat{d}$. Then the minor axis of the $\mathcal{S}_0$ ellipsoidal region can be maximized and admissible disturbance signal can be maximized by solving the following optimization problem:

$$\min_{W,M,L,Y,S,\eta,1/\beta} -\sigma_1 \eta + \sigma_2 \frac{1}{\beta}$$

$$\text{s.t} \sum_{\substack{k' \in \mathcal{K}(d) \\ k \succeq k'}} \frac{d!}{\pi(k')} \begin{bmatrix} \zeta_k \frac{1}{\beta} I & \zeta_k I \\ \star & W_{k-k'} \end{bmatrix} \geq 0,$$

$$l = 1, \ldots, m, \quad \forall k \in \mathcal{K}(g+d) \tag{27}$$

where the nonnegative scalars σ_1 and σ_2 are weighting factors that depend on the design needs. Then the controller can be constructed as $K_g(\alpha) = M_g(\alpha)[W_g(\alpha)]^{-1}$. As mentioned in [?], g (degree of the homogeneous polynomial Lyapunov matrix) and d (level of Pólya's relaxations) parameters can be arbitrarily chosen. However, the conservatism of the controller is highly related with these parameters. Hence, it can be shown that increasing the values of these parameters significantly reduces the conservatism of the controller.

5 SIMULATION and APPLICATION RESULTS

In this section, we demonstrate the benefits of our design on a inverted pendulum (IP) system via simulations and applications. The system is driven by a DC servo motor which is mounted on a cart where the cart can move on a horizontal linear rail system. The cart also carries a straight rod mounted by a free rotational joint. The objective is to design a controller which holds the rod perpendicular to the horizontal axis with minimum cart displacement. To overcome this problem, we present a novel controller design policy which minimizes the $\mathcal{L}_2$ gain from disturbance input to the controlled output while considering the actuator saturation. Here, we assumed that the disturbance signal acts on the cart as an additive inverse force and has square integrable energy.

The dynamic model of this system is obtained by using well-known Euler Lagrange mechanics which yields to

$$\ddot{x}_c = \frac{\begin{array}{c} g l_p^2 m_p^2 \cos(\varsigma)\sin(\varsigma) + (I_p + l_p^2 m_p)(F - w - B_{eq}\dot{x}_c(t)) \\ -l_p m_p (I_p + l_p^2 m_p)\sin(\varsigma)\dot{\varsigma}^2 + l_p m_p \cos(\varsigma)(-B_p \dot{\varsigma}) \end{array}}{(I_p + l_p^2 m_p)(m_p + m_c) - l_p^2 m_p^2 \cos(\varsigma)^2)} \tag{28}$$

$$\ddot{\varsigma} = \frac{\begin{array}{c} (m_p + m_c)(g l_p m_p \sin(\varsigma) - B_p\dot{\varsigma}) - l_p m_p \cos(\varsigma) \\ \times(-F + w + B_{eq}\dot{x}_c(t) + l_p m_p \sin(\varsigma)\dot{\varsigma}^2) \end{array}}{-(I_p + l_p^2 m_p)(m_p + m_c) - l_p^2 m_p^2 \cos(\varsigma)^2)} \tag{29}$$

Here, x_c stands for the cart displacement, ς is the angle of rod with respect to

Table 1: Physical quantities of the pendulum system.

Parametre	Aklama	Deer
m_c	Mass of the cart	0.7031kg
m_p	Mass of the pendulum	0.23kg
L_p	Length of the pendulum	0.6413m
I_p	Inertia of the pendulum	0.0078838kgm^2/sec^2
K_m	Back EMF constant of the DC motor	0.0077V.s/rad
K_t	Moment constant of the DC motor	0.0077V.s/rad
K_g	Gear ratio	3.71
R_a	Armature resistance of the DC motor	2.6Ω
r	Radius of the wheel of cart	0.0063m
B_{eq}	Damping ratio of the DC motor	4.3
B_p	Viscous damping ratio of the pendulum	0.0024

the vertical axis, w is the disturbance force applied to the cart and F is the applied force acting to the cart trough DC servomotor. The description of the other parameters used in the model and their parametric values are listed in Table 1.

In order to obtain the quasi-LPV model of the system, the time varying scheduling parameters that are online measurable or computable must be designated. In our problem, scheduling parameters might be selected as follows:

$$\begin{aligned}
\chi_1 =& \frac{\cos(\varsigma)\sin(\varsigma)}{(I_p + l_p^2 m_p)(m_p + m_c) - l_p^2 m_p^2 \cos(\varsigma)^2)\varsigma} \\
\chi_2 =& \frac{1}{(I_p + l_p^2 m_p)(m_p + m_c) - l_p^2 m_p^2 \cos(\varsigma)^2)} \\
\chi_3 =& \frac{\cos(\varsigma)}{(I_p + l_p^2 m_p)(m_p + m_c) - l_p^2 m_p^2 \cos(\varsigma)^2)} \\
\chi_4 =& \frac{\sin(\varsigma)\dot{\varsigma}}{(I_p + l_p^2 m_p)(m_p + m_c) - l_p^2 m_p^2 \cos(\varsigma)^2)} \\
\chi_5 =& \frac{\sin(\varsigma)}{(I_p + l_p^2 m_p)(m_p + m_c) - l_p^2 m_p^2 \cos(\varsigma)^2)\varsigma} \\
\chi_6 =& \frac{\cos(\varsigma)\sin\varsigma\dot{\varsigma}}{(I_p + l_p^2 m_p)(m_p + m_c) - l_p^2 m_p^2 \cos(\varsigma)^2)}
\end{aligned} \tag{30}$$

Consequently, with the definition $\chi := \begin{bmatrix} \chi_1, \dots, \chi_6 \end{bmatrix}^{\mathrm{T}}$, we get the quasi-LPV model

$$\dot{x} = A(\chi)x + B_w(\chi)w + B_u(\chi)u, \tag{31}$$

where

$$A(\chi) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & g l_p^2 m_p^2 \chi_1 & (-B_{eq} I_p - B_{eq} l_p^2 m_p)\chi_2 & -B_p l_p m_p \chi_3 - (I_p l_p m_p + l_p^3 m_p^2)\chi_4 \\ 0 & (g l_p m_p^2 + g l_p m_p m_c)\chi_5 & -B_{eq} l_p m_p \chi_3 & (-B_p m_p - B_p m_c)\chi_2 - l_p^2 m_p^2 \chi_6 \end{bmatrix}$$

$$B_w(\chi) = \begin{bmatrix} 0 \\ 0 \\ (-I_p - l_p^2 m_p)\chi_2 \\ -l_p m_p \chi_3 \end{bmatrix}, \quad x = \begin{bmatrix} x_c \\ \varsigma \\ \dot{x}_c \\ \dot{\varsigma} \end{bmatrix}, \tag{32}$$

$$B_u(\chi) = \begin{bmatrix} 0 \\ 0 \\ (I_p + l_p^2 m_p)\chi_2 \\ l_p m_p \chi_3 \end{bmatrix}, \qquad u = F$$

However, since the control force acting on the cart is provided by a DC servo motor, the dynamics of the motor also needs to be included into the overall dynamics. The motor used in this work has a voltage-force relation as

$$F = \frac{K_m K_g}{R_a r} V - \frac{K_m^2 K_g^2}{R_a r^2} \dot{x}_c \tag{33}$$

where V is the voltage applied to the armature of the motor where it has lower and upper hard limits $-13V$ and $+13V$, respectively. The other electrical parameters used in the model are also listed in Table 1.

When the force-voltage relation is included in the overall system dynamics, and the physical quantities listed in Table 1 are used, we obtain the quasi-LPV system matrices as follows:

$$A(\chi) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0.0565821\chi_1 & -0.396995\chi_2 & -0.00018227\chi_3 - 0.00250327\chi_4 \\ 0 & 0.69521\chi_5 & -0.914718\chi_3 & -0.00223951\chi_2 - 0.00576779\chi_6 \end{bmatrix}, \tag{34}$$

$$B_w(\chi) = \begin{bmatrix} 0 \\ 0 \\ -0.0329612\chi_2 \\ -0.075946\chi_3 \end{bmatrix}, \quad B_u(\chi) = \begin{bmatrix} 0 \\ 0 \\ 0.0569063\chi_2 \\ 0.131118\chi_3 \end{bmatrix}, \qquad (35)$$

For easiness, between the 6 candidate parameters discussed above, only the parameterχ_2 is selected as a scheduling parameter and the other scheduling parameters are assumed to be constant at their nominal values. Besides, $B_u(\chi)$ matrix is also assumed to be constant. When the pendulum is assumed to vary in between $[-30°, +30°]$, the vertices of the polytope are obtained as follows:

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2.2643 & -15.148 & -0.0073 \\ 0 & 27.8203 & -36.6044 & -0.0896 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2.2643 & -16.664 & -0.0073 \\ 0 & 27.8203 & -36.6044 & -0.0896 \end{bmatrix} \qquad (36)$$

$$B_{u_1} = \begin{bmatrix} 0 \\ 0 \\ 2.2772 \\ 5.2470 \end{bmatrix} \quad B_{u_2} = \begin{bmatrix} 0 \\ 0 \\ 2.2772 \\ 5.2470 \end{bmatrix} \qquad (37)$$

$$B_{w_1} = \begin{bmatrix} 0 \\ 0 \\ -1.2497 \\ -2 \end{bmatrix} \quad B_{w_2} = \begin{bmatrix} 0 \\ 0 \\ -1.3748 \\ -2 \end{bmatrix} \qquad (38)$$

$$z = Cx, \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \tag{39}$$

In order to demonstrate the efficiency of the proposed method, a set of simulations are performed. The simulations are carried on in three phases. In the first phase, the objective function (27) is used with the selection of weights $\sigma_1 = 0.02$ and $\sigma_2 = 30$. Besides, the polynomial degree g is selected as $g = 2$. In this phase, parameter d is varied from 0 up to 10. Figure 1 shows the variation of the objective function versus the parameter d. As can be observed, When d increases, the value of the objective function drastically decreases. Furthermore, the control effort noticeably decreases. To see the effect of the optimization on the controller performance, the controller also tried on the pendulum system for different d values. Consequently, the trajectories of pendulum angle $\varsigma(t)$ and cart position $x_c(t)$ are obtained as shown in Figure 2 and 3, respectively when the cart is exposed to a disturbance signal of the form

$$w(t) = 5[s(t-1) - s(t-2)] \, (\text{Volts}) \qquad \text{for } t \geq 0 \tag{40}$$

where $s(t)$ is a unit step function. Also Table 2, shows the 2-norm and the infinity norm of control signal used in the simulations. It is observed that the conservatism is highly related to the selection of the parameter d.

In the second phase, we solve the optimization problem (27) when η,and β are assigned as semi-definite programming variables. In this phase, the weighting factors are selected as $\sigma_1 = \sigma_2 = 1$. The aim of this phase is to demonstrate the influence of parameter g on the volume of the set $\mathcal{S}_0 = \mathcal{E}(P(\alpha), \beta) = x \in \mathbb{R}^n :$

$x^T W(\alpha)^{-1} x \leq \beta$ and on the performance of the controller. Figure 4 shows the ellipsoidal admissible projection sets of $\mathcal{S}_0$ for different values of g. Note that these projections are drawn for $\alpha_1 = \alpha_2 = 0.5$. Again, when g increases, the set enlarges remarkably. In our application, it is observed that the ellipsoids converge when $g \geq 5$. In order to demonstrate the sensitivity of the controller against g, the variation of pendulum angle and cart position are also plotted in Figure 5 and Figure 6, respectively when the cart is exposed to an additive disturbance signal as given in (40). Furthermore, Table 3 shows the 2-norm and the infinity norm of control signal used in the simulations. As in the previous case, we found out that the control effort noticeably influenced by the increasing precision of g.

In the last phase of the simulations, we aimed to show the influence of g and d parameters on the conservatism of the controller through the optimization of parameter η when $1/\beta = 11.2$. Table 4 shows the variation of η with respect to different combinations of g and d values. Note that, although for some small values of g and d fail to provide a feasible solution, with increasing g and d values, the solution of the problem appears to be exists. This result obviously shows the efficiency of the design policy.

Alongside of simulations, the efficiency of the proposed method also examined on an experimental inverted pendulum system in laboratory. During the tests, η and β are assigned as semi-definite programming variables in the optimization problem (27). Also, the weighting factors are selected as $\sigma_1 = \sigma_2 = 1$. Respectively, Figure 7 and Figure 8 shows the variation of the pendulum angle

$\varsigma(t)$ and cart position $x_c(t)$ when the cart is exposed to a disturbance

$$w(t) = 2[u(t-3) - u(t-4)]\,(\text{Volts}) \qquad \text{for } t \geq 0 \tag{41}$$

Observe that the controller performance gradually increases with the raise of g and d which confirm the results obtained in simulations. To show the benefits of the proposed method on the controller energy, both the infinity and 2-norms of control signal $u(t)$ are also recorded. Table 5 shows the computed 2-norm and infinity norm of $u(t)$ for different combinations of g and d. Observe that, increase in the precision of the controller i.e., increase in the values of g and d, drastically reduces the energy consumption of the controller.

6 CONCLUSION

In this paper, a new design method for $\mathcal{L}_2$ gain-scheduling nonlinear state feedback controller for LPV systems having actuator saturations has been addressed by using parameter dependent Lyapunov functions. The method provides a systematic procedure to generate a sequence of LMI conditions of increasing precision for obtaining a sub-optimal, $\mathcal{L}_2$ gain, state-feedback controllers. The obtained LMI relaxations are parameterized on the degree of the homogeneous polynomial (g) and the Pólya's relaxation level (d). At the end, the validation and the efficiency of the proposed method has been demonstrated on an inverted pendulum system both through simulations and experiments. Both the applications and experiments showed that increase in the precision of the controller parameters g and d, provides remarkable raise in the performance of the controller.

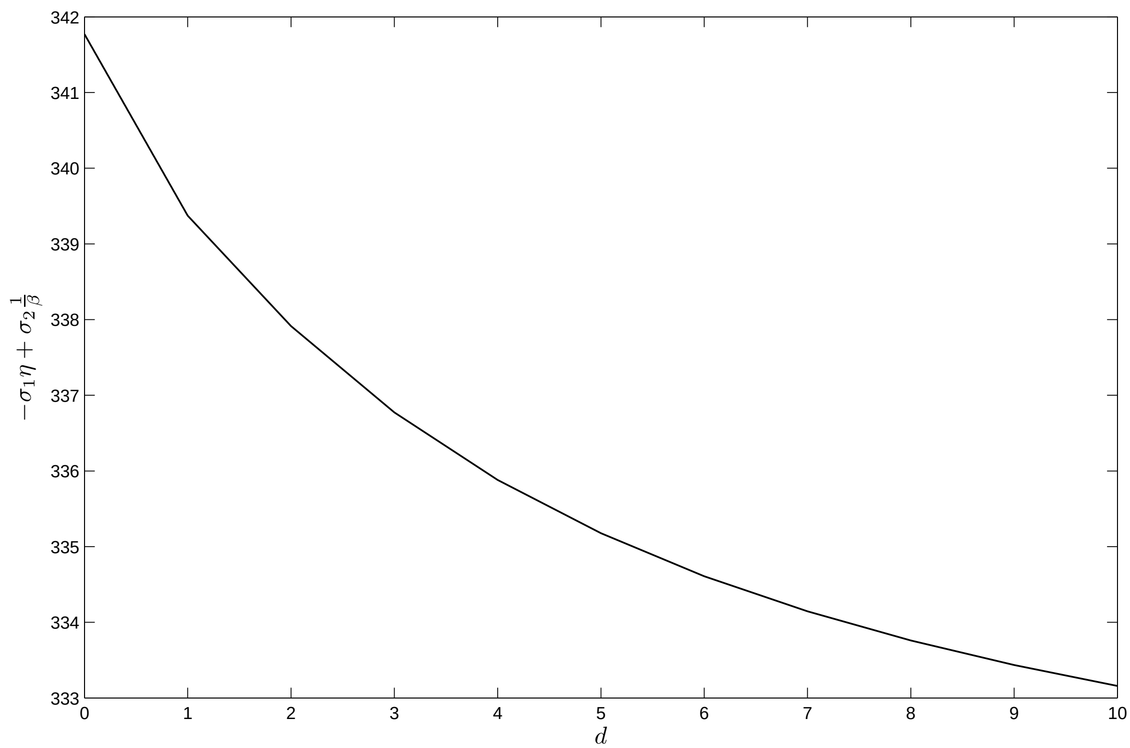

Figure 1: The variation of $-\sigma_1\eta + \sigma_2\frac{1}{\beta}$ when $g = 2$.

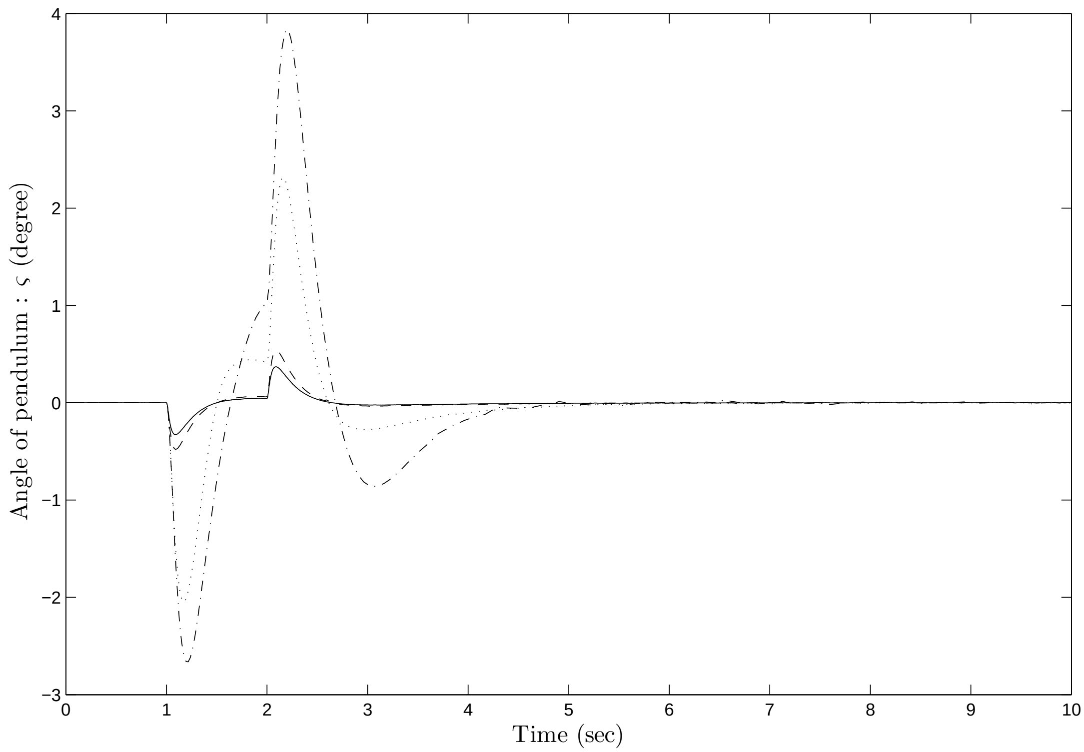

Figure 2: The variation of $\varsigma(t)$ when $g = 2$, $d = 0$ (dashed-dot), $d = 1$ (dotted), $d = 4$ (dashed), $d = 10$ (solid).

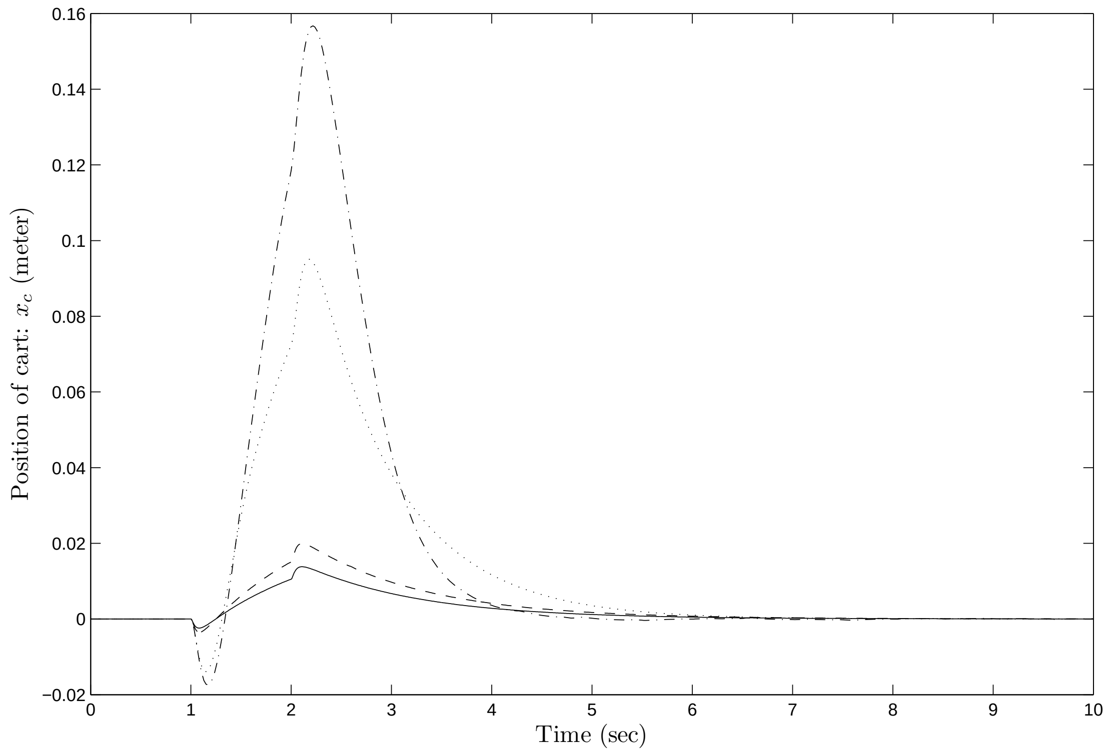

Figure 3: The variation of $x_c(t)$ when $g = 2$, $d = 0$ (dashed-dot), $d = 1$ (dotted), $d = 4$ (dashed), $d = 10$ (solid).

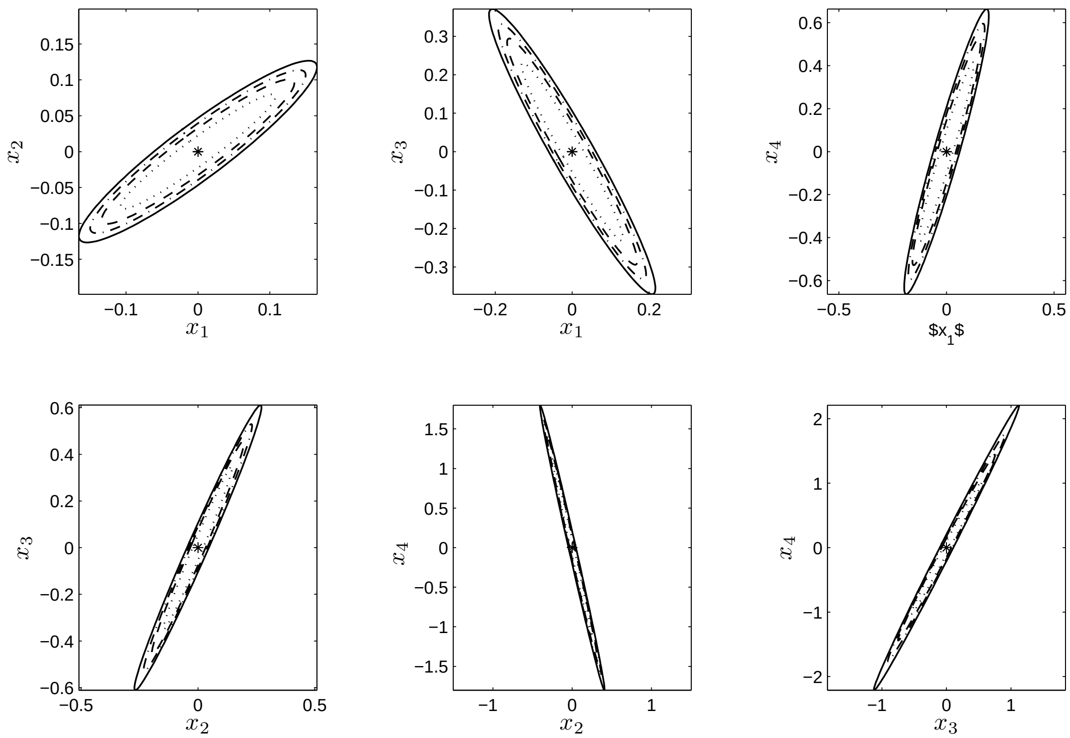

Figure 4: Ellipsoidal admissible projection sets of $\mathcal{S}_0$ for different values of g:

$g = 1$(dotted), $g = 2$(dashed), $g = 4$(dashed-dot), $g = 10$(solid)

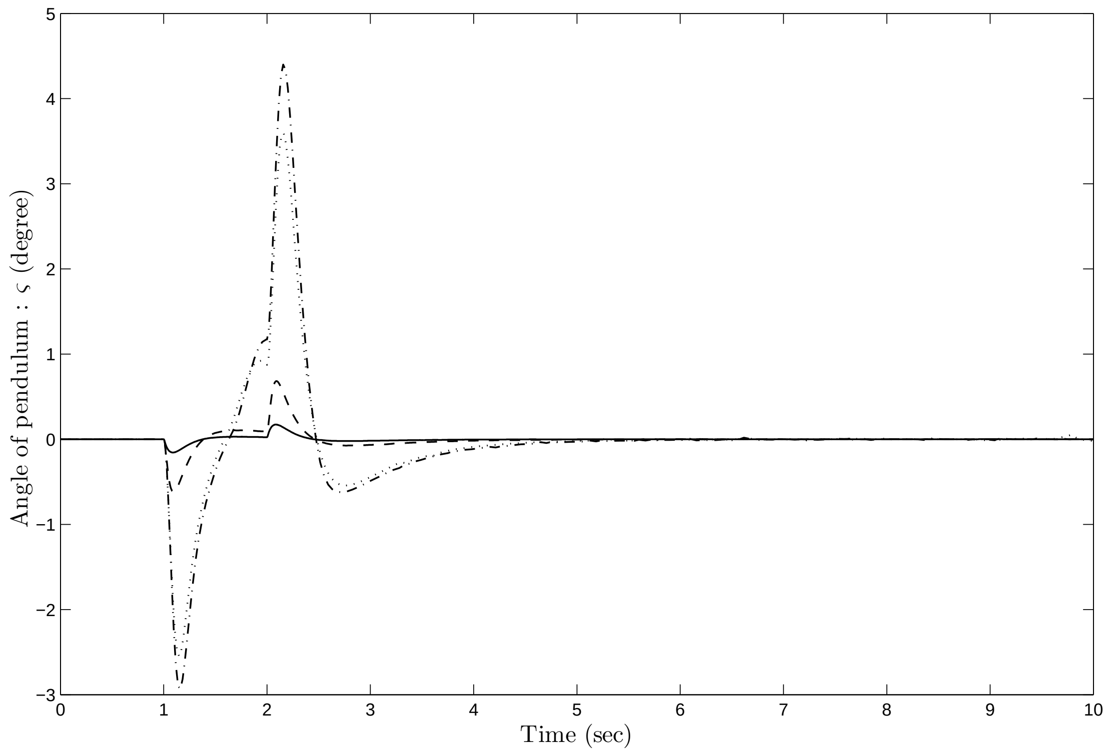

Figure 5: The variation of $\varsigma(t)$ when $d = 0$, $g = 1$ (dashed-dot), $g = 2$ (dotted), $g = 5$ (dashed), $g = 10$ (solid).

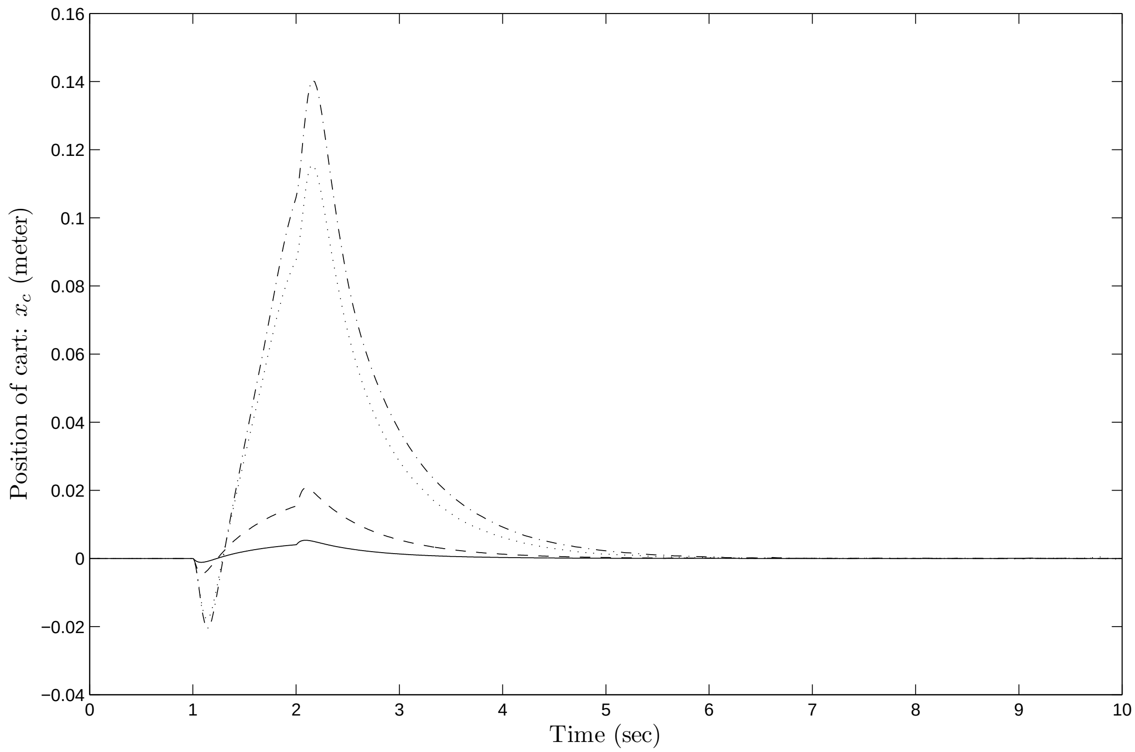

Figure 6: The variation of $x_c(t)$ when $d = 0$, $g = 1$ (dashed-dot), $g = 2$ (dotted), $g = 5$ (dashed), $g = 10$ (solid).

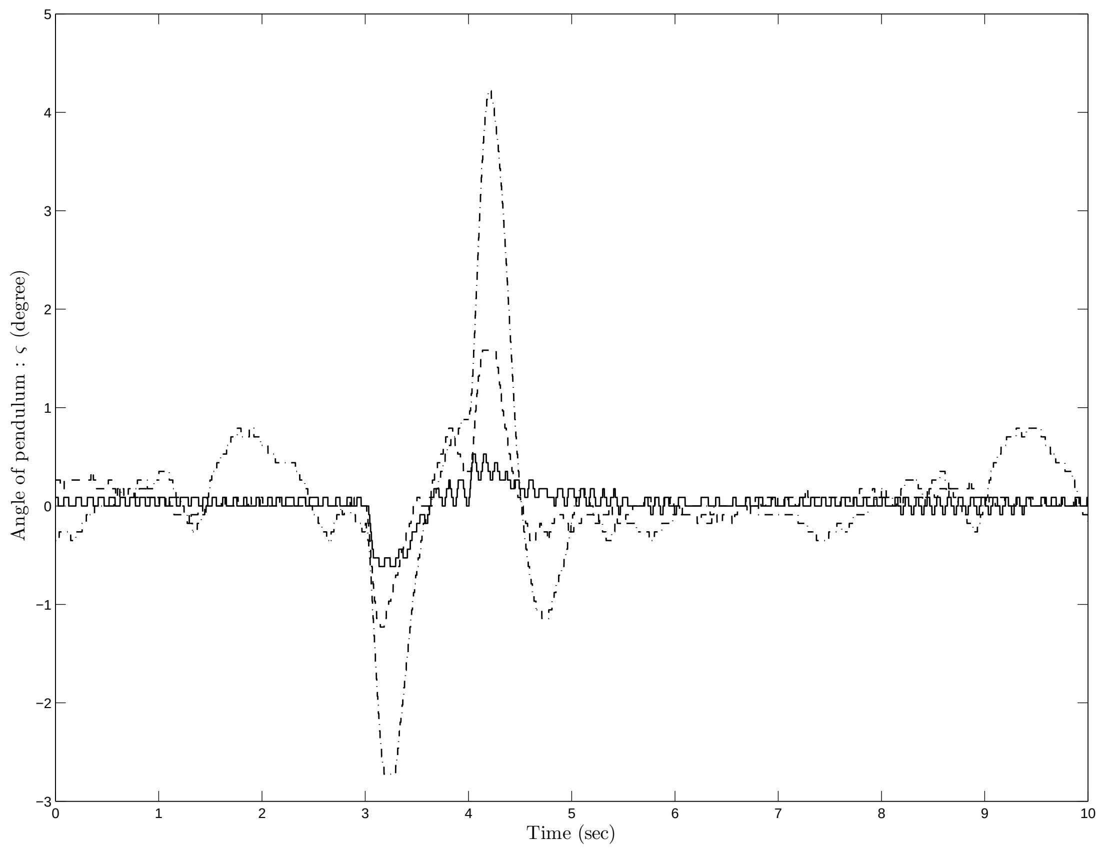

Figure 7: The application results for the variation of $\varsigma(t)$ when [$d = 0$, $g = 1$ (dashed-dot)], [$d = 0$, $g = 3$ (dashed)], [$d = 6$, $g = 6$ (solid)].

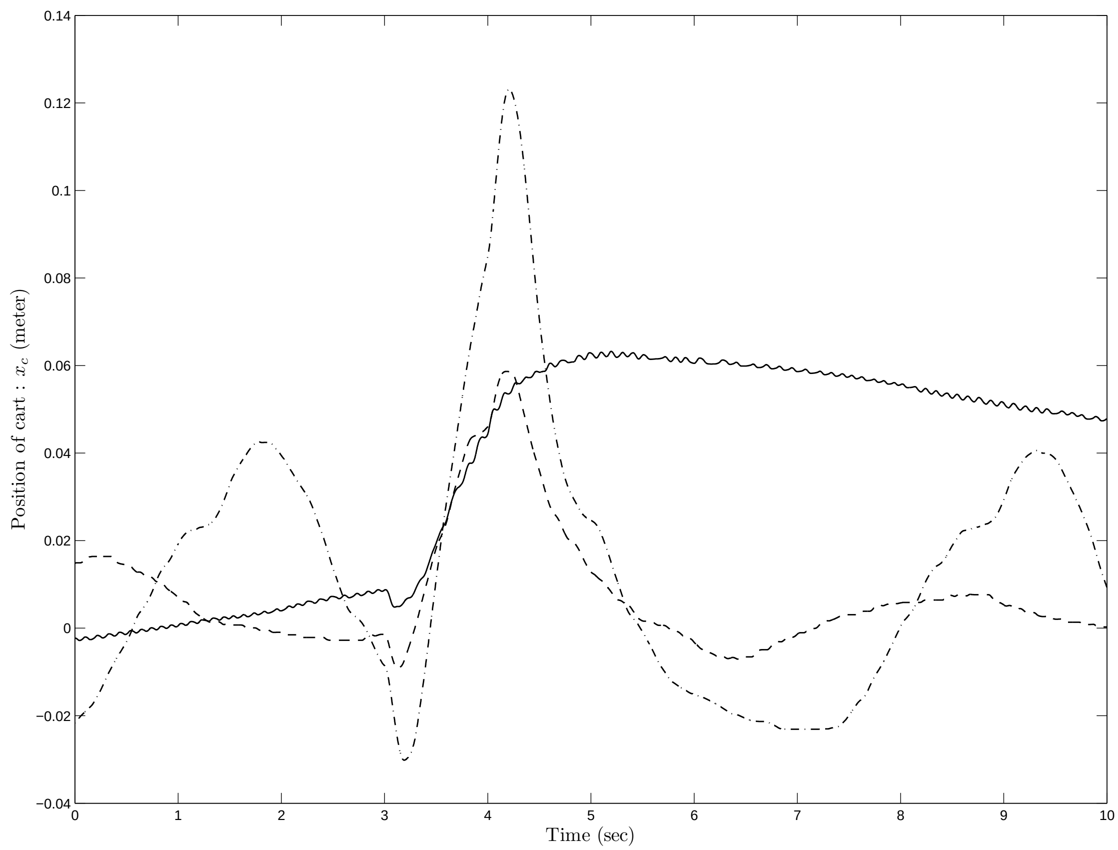

Figure 8: The application results for the variation of $x_c(t)$ when [$d = 0$, $g = 1$ (dashed-dot)], [$d = 0$, $g = 3$ (dashed)], [$d = 6$, $g = 6$ (solid)].

d	0	1	4	10
2-Norm of control signal	13.0247	11.6928	9.6157	9.2439
inf-Norm of control signal	4.3288	4.0868	3.3014	3.1569

Table 2: Norms of control signals for different values of d.

g	1	2	5	10
2-Norm of control signal	13.0538	12.2292	9.7750	9.2190
inf-Norm of control signal	4.6519	4.3153	3.6233	3.0795

Table 3: Norms of control signals for different values of g.

<table>
<tr><th></th><th colspan="6">d</th></tr>
<tr><th>g</th><th>0</th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th></tr>
<tr><td>1</td><td colspan="6">Infeasible</td></tr>
<tr><td>2</td><td colspan="4">Infeasible</td><td>17.6891</td><td>18.8079</td></tr>
<tr><td>3</td><td colspan="2">Infeasible</td><td>17.5920</td><td>19.0073</td><td>19.5819</td><td>19.9569</td></tr>
<tr><td>4</td><td>15.0845</td><td>18.1243</td><td>19.0476</td><td>19.8363</td><td>20.2056</td><td>20.4737</td></tr>
<tr><td>5</td><td>18.4270</td><td>19.2582</td><td>19.8118</td><td>20.1386</td><td>20.4872</td><td>20.7810</td></tr>
<tr><td>6</td><td>19.3303</td><td>19.9132</td><td>20.2149</td><td>20.5735</td><td>20.6823</td><td>21.1416</td></tr>
</table>

Table 4: η values for different values of g and d.

	g=1 d=0	g=3 d=0	g=6 d=6
2-Norm of control signal	137.6668	104.5607	101.7835
inf-Norm of control signal	4.5016	4.0944	4.0734

Table 5: Norms of control signals for different values of g and d.